UNIT- IV

VECTOR DIFFERENTIATION

vector point function, scalar point function, gradient, divergents and curl.

Directional derivatives, tangent plane and normal line and vector identities, scalar potential functions, solenoidal and irrotational vectors.

Scalar: A physical quantity which is having only magnitude is called a scalar.

Ex: Distance, length, Area, volume, Time, Temperature, speed etc.

Scalar point function: In 3 dimensional space a each point $P(x,y,z)$ a unique real no. ϕ is mapped then $\phi(x,y,z)$ is called a scalar point function.

Ex: In a heated solid at each point $P(x,y,z)$ there will be Temp.$(T) \rightarrow (x,y,z)$ then T is called a scalar point function.

vector: A physical quantity which has both magnitude and direction is known as a vector.

Ex: velocity, Acceleration, force, weight, displacement etc.

Vector point function: To each point $P(x,y,z)$ a unique vector $F(x,y,z)$ is mapped then $\vec{F}$ is called a vector point function.

Ex: At each point $P(x,y,z)$ on its path the particle having velocity $\overline{V}$ is a vector point function.

Generally, the vectors are denoted by $\overline{F}$ (or) $\vec{F}$ and it is denoted by $\overline{F} = \vec{F} = F_1\overline{i} + F_2\overline{j} + F_3\overline{K}$. Here f_1, f_2, f_3 are components of a vector and i, j, K are unit vectors along

x, y, z axes.

Vector Differential operator: It is denoted by ∇ and it is defined as $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

Position vector: Let $P(x, y, z)$ be any point in the space then the position vector of P w.r.t the origin $O(0,0,0)$ is denoted by $\overline{OP} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

The magnitude of the position vector $\bar{r}$ can be denoted by $|\bar{r}|$ and it is denoted by $|\bar{r}| = |x\bar{i} + y\bar{j} + z\bar{k}| = \sqrt{x^2+y^2+z^2}$

Diffn $|\bar{r}|$ w.r.t 'x' on b/s $\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \frac{x}{|\bar{r}|} = \frac{x}{r}$$

Illy Diffn w.r.t y and z $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

i. Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ be the dot product (or) scalar product of $\bar{a}.\bar{b} = (a_1\bar{i}+a_2\bar{j}+a_3\bar{k}).(b_1\bar{i}+b_2\bar{j}+b_3\bar{k})$

$$\Rightarrow \bar{a}.\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$i.i = 1$

$i.j = 0$

ii. The cross product (or) vector product of $\bar{a}$ and $\bar{b}$ is denoted by: $\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

iii. If θ is the angle b/w the vectors $\bar{a}$ and $\bar{b}$ then $\cos\theta = \frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}$

iv. If the vectors $\bar{a}$ and $\bar{b}$ are $\perp^r$ then dot product $\bar{a}.\bar{b} = 0$.

v. Gradient of Scalar point function.

If $\phi(x, y, z)$ is continuous differentiable scalar point fun ction then Gradient of ϕ and it is denoted by Grad ϕ (or) $\nabla\phi$ and it is defined as Grad $\phi = \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

properties:

i. The gradient of a scalar point function is a vector.

ii. If f and g are two scalar point functions then grad(f±g) (or) $\nabla(f \pm g)$ = grad f ± grad g

iii. $\nabla(fg) = f(\text{grad } g) + g(\text{grad } f)$

$= f(\nabla g) + g(\nabla f)$

Normal vector: Normal vector of a scalar point function if ϕ is any scalar point function then $\nabla\phi$ is called Normal vector of ϕ

$|\nabla\phi|$ $\quad \dfrac{\nabla\phi}{|\nabla\phi|}$ is a unit vector.

The magnitude of Normal vector is $\nabla\phi$.

Angle of intersection:

Angle b/w two surfaces:

The angle b/w two surfaces $\phi_1(x,y,z)$ and $\phi_2(x,y,z)$ is

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| \cdot |\bar{n}_2|}$$

i. where $\bar{n}_1$ is a normal vector of $\phi_1(x,y,z)$ & n_2 is a normal vector of $\phi_2(x,y,z)$ at point $P(x,y,z)$.

ii. Let $\bar{n}_1$ be the normal vector to the surface $\phi_1(x,y,z)$ at the point $P_1(x,y,z)$. $\bar{n}_2$ be the normal vector to the surface $\phi_2(x,y,z)$ at the point $P_2(x,y,z)$. Then the angle b/w 2 normals is $\cos\theta = \dfrac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| \cdot |\bar{n}_2|}$

iii. If two surfaces $\phi_1(x,y,z)$ and $\phi_2(x,y,z)$ intersects orthogonally $\nabla\phi_1 \cdot \nabla\phi_2 = 0$.

Find the Normal vector to the surface and magnitude of the surface $\phi = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ and find the normal vector.

Given that $\phi = x^2yz + 4xz^2$ and

normal vector $\bar{n} = \text{Grad}\,\phi = \nabla\phi$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \bar{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \bar{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= \bar{i}(2xyz + 4z^2) + \bar{j}(x^2z + 0) + \bar{k}(x^2y + 8xz)$$

$$\bar{n} = (\nabla\phi)_{(1,-2,-1)} = \bar{i}(4+4) + \bar{j}(-1) + \bar{k}(-2-8)$$

$$= 8\bar{i} - \bar{j} - 10\bar{k} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$|\bar{n}| = \sqrt{x^2+y^2+z^2} = \sqrt{(8)^2 + (-1)^2 + (-10)^2} = \sqrt{165}$$

Unit Normal vector (U.N.V) $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{8\bar{i} - \bar{j} - 10\bar{k}}{\sqrt{165}}$

Find Grad ϕ at the point $(1,0,0)$ & $|\nabla\phi|$ where $\phi = x^2+y+z-1$.

Given that $\phi = x^2+y+z-1$

$$\bar{n} = \text{Grad}\,\phi = \nabla\phi$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(x^2+y+z-1) + \bar{j}\frac{\partial}{\partial y}(x^2+y+z-1) + \bar{k}\frac{\partial}{\partial z}(x^2+y+z-1)$$

$$= \bar{i}(2x) + \bar{j}(1) + \bar{k}(1)$$

$$= 2x\bar{i} + \bar{j} + \bar{k}$$

$$\bar{n} = (\nabla\phi)_{(1,0,0)} = 2\bar{i}$$

$$|\bar{n}| = \sqrt{x^2+y^2+z^2} = \sqrt{(2)^2} = 2$$

Find unit Normal vector to the surface $x^2y+2zx=4$ at the point $(2,-2,3)$

Given that $\phi = x^2y+2zx-4$ and

$$\bar{n} = \text{Grad}\,\phi = \nabla\phi$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(x^2y+2zx-4) + \bar{j}\frac{\partial}{\partial y}(x^2y+2zx-4) + \bar{k}\frac{\partial}{\partial z}(x^2y+2zx-4)$$

$$= \bar{i}(2xy+2z) + \bar{j}(x^2+0) + \bar{k}(0+2x)$$

$$= (2xy+2z)\bar{i} + x^2\bar{j} + 2x\bar{k}$$

$$\bar{n} = (\nabla\phi)\underset{x\ \ y\ \ z}{(2,-2,3)} = \bar{i}(-8+6) + 4\bar{j} + 4\bar{k}$$

$$= -2\bar{i} + 4\bar{j} + 4\bar{k}$$

$$|\bar{n}| = \sqrt{x^2+y^2+z^2} = \sqrt{(-2)^2+(4)^2+(4)^2} = \sqrt{36} = 6$$

unit Normal vector $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{-2\bar{i}+4\bar{j}+4\bar{k}}{6}$

Find the unit normal vector of $x^2+y^2 = z$ at the point $(-1,-2,5)$

Given that $\phi = x^2+y^2-z$ and

$$\bar{n} = \text{Grad}\,\phi = \nabla\phi$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(x^2+y^2-z) + \bar{j}\frac{\partial}{\partial y}(x^2+y^2-z) + \bar{k}(x^2+y^2-z)$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(-1) = 2x\bar{i} + 2y\bar{j} - \bar{k}$$

$$\bar{n} = (\nabla\phi)(-1,-2,5) = \bar{i}(-2) + \bar{j}(-4) + (-1)\bar{k}$$

$$= -2\bar{i} - 4\bar{j} - \bar{k}$$

$$|\bar{n}| = \sqrt{x^2+y^2+z^2} = \sqrt{(-2)^2+(-4)^2+(-1)^2} = \sqrt{21} = 2\sqrt{5}\,\sqrt{21}$$

Unit Normal vector $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{-2\bar{i}-4\bar{j}-\bar{k}}{2\sqrt{5}\,\sqrt{21}}$

Find the angle b/w two normals $x^2 = yz$ at $(1,1,1)$ and $(2,4,1)$

G·let $\phi = x^2 - yz$

let n_1 be the normal vector at $(1,1,1)$

$$\bar{n}_1 = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2x-0) + \bar{j}(0-z) + \bar{k}(0-y)$$

$$= 2x\bar{i} - z\bar{j} - y\bar{k}$$

$$\bar{n}_1 = \nabla\phi\,(1,1,1) = 2\bar{i} - \bar{j} - \bar{k} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$|\bar{n}_1| = |\nabla\phi| = \sqrt{x^2+y^2+z^2} = \sqrt{(2)^2+(-1)^2+(-1)^2} = \sqrt{6}$$

let n_2 be the normal vector at $(2,4,1)$

$$\bar{n}_2 = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{z}\frac{\partial\phi}{\partial z}$$

$$= 2x\bar{i} - z\bar{j} - y\bar{k}$$

$$\bar{n}_2 = (\nabla\phi)\,(\underset{x}{2}, \underset{y}{4}, \underset{z}{1}) = 4\bar{i} - \bar{j} - 4\bar{k}$$

$$= x\bar{i} + y\bar{j} + z\bar{k}$$

$$|\bar{n}_2| = |\nabla\phi| = \sqrt{x^2+y^2+z^2} = \sqrt{(4)^2+(-1)^2+(-4)^2} = \sqrt{33}$$

The Angle b/w two normals is $\cos\theta = \dfrac{\bar{n}_1 . \bar{n}_2}{|\bar{n}_1| . |\bar{n}_2|}$

$$\cos\theta = \frac{(2\bar{i}-\bar{j}-\bar{k}).(4\bar{i}-\bar{j}-4\bar{k})}{\sqrt{6}.\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}.\sqrt{33}} = \frac{13}{\sqrt{6}.\sqrt{33}}$$

$$\theta = \cos^{-1}\frac{13}{\sqrt{6}.\sqrt{33}}$$

Find the angle b/w two surfaces $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$

Let $\phi_1 = x^2+y^2+z^2-29$ and

$$\phi_2 = x^2+y^2+z^2+4x-6y-8z-47$$

let the normal vector $\bar{n}_1 = \nabla\phi_1$

$$= \bar{i}\frac{\partial\phi_1}{\partial x} + \bar{j}\frac{\partial\phi_1}{\partial y} + \bar{k}\frac{\partial\phi_1}{\partial z}$$

$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z) = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$(\bar{n}_1) = (\nabla\phi_1)(4, -3, 2) = 8\bar{i} - 6\bar{j} + 4\bar{k} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{n}_1| = |\nabla\phi| = \sqrt{x^2+y^2+z^2} = \sqrt{(8)^2+(-6)^2+(4)^2} = \sqrt{116}$

let the normal vector $\bar{n}_2 = \nabla\phi_2$

$= \bar{i}\frac{\partial\phi_2}{\partial x} + \bar{j}\frac{\partial\phi_2}{\partial y} + \bar{k}\frac{\partial\phi_2}{\partial z}$

$= \bar{i}(2x+4) + \bar{j}(2y-6) + \bar{k}(2z-8)$

$= (2x+4)\bar{i} + (2y-6)\bar{j} + (2z-8)\bar{k}$

$(\bar{n}_1) = (\nabla\phi_2)(4, -3, 2) = (8+4)\bar{i} + (-6-6)\bar{j} + (4-8)\bar{k}$

$= 12\bar{i} - 12\bar{j} - 4\bar{k}$

$|\bar{n}_2| = |\nabla\phi| = \sqrt{x^2+y^2+z^2} = \sqrt{(12)^2+(-12)^2+(-4)^2} = \sqrt{304}$

Angle b/w two surfaces (or) two normals

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| \cdot |\bar{n}_2|}$$

$$= \frac{(8\bar{i}-6\bar{j}+4\bar{k}) \cdot (12\bar{i}-12\bar{j}-4\bar{k})}{\sqrt{116} \cdot \sqrt{304}}$$

$$= \frac{96+72-16}{\sqrt{116} \cdot \sqrt{304}} = \frac{152}{\sqrt{116} \cdot \sqrt{304}}$$

$$\cos\theta = \frac{152}{\sqrt{116} \cdot \sqrt{304}}$$

$$\theta = \cos^{-1}\left(\frac{152}{\sqrt{116} \cdot \sqrt{304}}\right)$$

Find the values of λ, μ so that the surface $\lambda x^2 - \mu yz - (\lambda+2)x$ and $4x^2y + z^3 = 4$ intersect orthogonally at the point $(1, -1, 2)$

let $\phi_1(x, y, z)$ is $\lambda x^2 - \mu yz - (\lambda+2)x$

$\phi_2(x, y, z) = 4x^2y + z^3 - 4$

Given that the two surfaces are intersecting at the point $(1,-1,2)$ and substitute $P(x,y,z)=P(1,-1,2)$ in ①

$$\lambda x^2-\mu yz-(\lambda+2)x=0$$

$$\lambda(1)^2-\mu(-1)(2)-(\lambda+2)1=0$$

$$\Rightarrow \lambda+2\mu-\lambda-2=0$$

$$\Rightarrow 2\mu=2 \Rightarrow \mu=1$$

$$\bar{n}_1=\nabla\phi_1=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\lambda x^2-\mu yz-(\lambda+2)x$$

$$=\bar{i}(2\lambda x-(\lambda+2))+\bar{j}(-\mu z)+\bar{k}(-\mu y)$$

$$=\lambda\bar{i}(2x\cancel{\lambda}(\lambda-2)\mp\bar{j}(\mu z)-\bar{k}(\mu y))$$

$$\bar{n}_1=\nabla\phi=(\underset{x}{1},\underset{y}{-1},\underset{z}{2})=\bar{i}(2\lambda-\lambda-2)-\bar{j}(2\mu z)+\bar{k}\mu$$

$$=\bar{i}(\lambda-2)-2\mu\bar{j}+\bar{k}$$

$$\bar{n}_2=\nabla\phi_2=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(4x^2y+z^3-4)$$

$$=\bar{i}(8xy)+\bar{j}(4x^2.1)+\bar{k}(3z^2)$$

$$\bar{n}_2=\nabla\phi_2=(\underset{x}{1},\underset{y}{-1},\underset{z}{2})=-8\bar{i}+4\bar{j}+12\bar{k}$$

$$\nabla\phi_1\cdot\nabla\phi_2=0 \Rightarrow \left[(\lambda-2)i-2\mu\bar{j}+\bar{k}\right]\cdot(-8i+4\bar{j}+12\bar{k})=0$$

$$\Rightarrow -8\lambda+16-8\mu+12=0$$

$$\Rightarrow -8\lambda+16-8(1)+12=0$$

$$\Rightarrow -8\lambda+16-8+12=0$$

$$\Rightarrow -8\lambda+20=0 \Rightarrow 20=8\lambda \Rightarrow \lambda=\frac{5}{2}$$

$\therefore$ The two surfaces intersect orthogonally.

Find the angle b/w two surfaces $x^2yz=3x+z^2$ and $3x^2-y^2+2z=1$ at the point $(1,-2,1)$

If $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$, $r=|\bar{r}|$ then (i) $\nabla f(r)=\frac{f'(r)}{r}\cdot\bar{r}$

ii) $\nabla r^n=nr^{n-2}\cdot\bar{r}$

iii) $\nabla(\log r)=r^{-2}\cdot\bar{r}$

iv) $\nabla \frac{1}{r} = \frac{-1}{r^3}\bar{r}$

v) $\nabla r = \frac{\bar{r}}{r}$

$\bar{r} = xi + yj + zk$ — ①

Given that $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ — ①

squ on b/s $\Rightarrow r^2 = x^2+y^2+z^2$ — ②

Diff w.r.t 'x' ②

$$2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Diff w.r.t 'y'

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Diff. w.r.t 'z'

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

i) $\nabla f(r) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$

$$= \bar{i} f'(r)\frac{\partial r}{\partial x} + \bar{j} f'(r)\frac{\partial r}{\partial y} + \bar{k} f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left(\bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}\right)$$

$$= f'(r)\left(\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right)$$

$$= \frac{f'(r)}{r}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= \frac{f'(r)}{r} \cdot \bar{r}$$

ii) $\nabla r^n = \left(\bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$

$$= \bar{i} \cdot nr^{n-1}\frac{\partial r}{\partial x} + \bar{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \bar{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(\bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}\right)$$

$$= nr^{n-1}\left(\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right)$$

$$= \frac{n \cdot r^{n-1}}{r} (x\bar{i} + y\bar{j} + z\bar{k})$$

$$= n r^{n-2} \cdot \bar{r}$$

iii) $$\nabla(\log r) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(\log r)$$

$$= \left(\bar{i}\frac{1}{r}\frac{\partial r}{\partial x} + \bar{j}\frac{1}{r}\frac{\partial r}{\partial y} + \bar{k}\frac{1}{r}\frac{\partial r}{\partial z}\right)$$

$$= \frac{1}{r}\left(\bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}\right)$$

$$= \frac{1}{r}\left(\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right)$$

$$= \frac{1}{r^2}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= \bar{r}^{2} \cdot \bar{r}$$

iv) $$\nabla\left(\frac{1}{r}\right) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \bar{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \bar{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \bar{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \frac{-1}{r^2}\left(\bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left(\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right)$$

$$= \frac{-1}{r^3}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= \frac{-1}{r^3} \cdot \bar{r}$$

v) $$\nabla r = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(r)$$

$$= \bar{i}(1)\frac{\partial r}{\partial x} + \bar{j}(1)\frac{\partial r}{\partial y} + \bar{k}(1)\frac{\partial r}{\partial z}$$

$$= \left(\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right)$$

$$= \frac{1}{r}(x\bar{i} + y\bar{j} + z\bar{k}) = \frac{\bar{r}}{r}$$

Find the angle b/w two surfaces $x^2yz = 3x + z^2$ and $3x^2 - y^2 +$ $2z = 1$ at the point $(1, -2, 1)$

Let $\phi_1 = x^2yz - 3x - z^2$ and

$\phi_2 = 3x^2 - y^2 + 2z - 1$

let the normal vector $\bar{n}_1 = \nabla\phi_1$

$= \bar{i}\dfrac{\partial\phi_1}{\partial x} + \bar{j}\dfrac{\partial\phi_1}{\partial y} + \bar{k}\dfrac{\partial\phi_1}{\partial z}$

$= \bar{i}(2xyz - 3) + j(\overset{x^2z}{0}) + \bar{k}(-2z-)(x^2y - 2z)$

$= (2xyz - 3)\bar{i} + x^2z\,\bar{j} + 2z\,\bar{k}(-2-2)$

$\bar{n}_1 = \nabla\phi_1 \underset{x\ y\ z}{(1, -2, 1)} = -7\bar{i} + \bar{j} - 4\bar{k}$

$|\bar{n}_1| = \nabla\phi_1 = \sqrt{x^2 + y^2 + z^2} = \sqrt{49 + 1 + 46} = \sqrt{66}$

let the normal vector $\bar{n}_2 = \nabla\phi_2$

$= \bar{i}\dfrac{\partial\phi_2}{\partial x} + \bar{j}\dfrac{\partial\phi_2}{\partial y} + \bar{k}\dfrac{\partial\phi_2}{\partial z}$

$= \bar{i}(6x) + j(-2y) + \bar{k}(2)$

$= 6x\bar{i} - 2y\bar{j} + 2\bar{k}$

$\bar{n}_1 = \nabla\phi_2 (1, -2, 1) = 6\bar{i} + 4\bar{j} + 2\bar{k}$

$|\bar{n}_2| = \nabla\phi_2 = \sqrt{x^2 + y^2 + z^2} = \sqrt{36 + 16 + 4} = \sqrt{56}$

Angle b/w two surfaces $\cos\theta = \dfrac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| \cdot |\bar{n}_2|}$

$\cos\theta = \dfrac{(-7\bar{i} + \bar{j} - 4\bar{k})(6\bar{i} + 4\bar{j} + 2\bar{k})}{\sqrt{54} \cdot \sqrt{56}}$

$= \dfrac{-42 + 4 - 8}{\sqrt{66} \cdot \sqrt{56}}$

$\cos\theta = \dfrac{-46}{\sqrt{56} \cdot \sqrt{56}}$

$\theta = \cos^{-1}\left(\dfrac{-42}{\sqrt{56}\sqrt{56}}\right)$

Directional Derivative : Let $\phi(x,y,z)$ be any scalar point function in the region R and $P(x,y,z)$ be any point in that region. suppose $Q(x,y,z)$ is a point in this region is the neighbourhood of the P in the direction of a vector $\vec{a}$. If

$$\lim_{Q\to P} \frac{\phi(Q)-\phi(P)}{OQ-OP}$$ exist. Then it is called the directional derivative of ϕ at the point P in the direction of $\vec{a}$.

i. The directional derivative of scalar point function $\phi(x,y,z)$ at the point $P(x,y,z)$ in the direction of $\vec{a}$ is $\vec{E} = \nabla\phi.\bar{e} = \nabla\phi.\frac{\bar{a}}{|\bar{a}|}$.

where $\bar{e}$ is unit Normal vector of $\frac{\bar{a}}{|\bar{a}|}$.

ii. The max. value of directional derivative is $|\vec{E}| = |\nabla\phi.\bar{e}| = |\nabla\phi|.1$

$$|\vec{E}| = \nabla\phi$$

Find the directional derivative of $\phi = 3x^2+2y-3z$ at the point $(1,1,1)$ in the direction of $2\bar{i}+2\bar{j}-\bar{k}$ also find the max. value of directional derivative.

Given that the scalar point function $\phi = 3x^2+2y-3z$ and let $P(x,y,z) = P(1,1,1)$ and vector $\vec{a} = 2\bar{i}+2\bar{j}-\bar{k}$

$$\vec{E} = \nabla\phi.\vec{e} = \nabla\phi.\frac{\bar{a}}{|\bar{a}|}$$

$$\bar{n} = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(3x^2+2y-3z) + \bar{j}\frac{\partial}{\partial y}(3x^2+2y-3z) + \bar{k}\frac{\partial}{\partial z}(3x^2+2y-3z)$$

$$= \bar{i}(6x) + \bar{j}(2) + \bar{k}(-3)$$

$$= 6x\bar{i} + 2\bar{j} - 3\bar{k}$$

$$\bar{n}_1\ (\nabla\phi)P(x,y,z) = (\nabla\phi)_{(1,1,1)} = 6\bar{i}+2\bar{j}-3\bar{k}$$

Unit vector normal vector $\vec{e} = \frac{\bar{a}}{|\bar{a}|}$

$$\Rightarrow \frac{\bar{a}}{|\bar{a}|} = \frac{2\bar{i}+2\bar{j}-\bar{k}}{\sqrt{(2)^2+(2)^2+(-1)^2}}$$

$$= \frac{2\bar{i}+2\bar{j}-\bar{k}}{3}$$

The directional derivative $\vec{E} = \nabla\phi.\vec{e} = \nabla\phi.\frac{\bar{a}}{|\bar{a}|}$

$$\vec{E} = (6\bar{i}+2\bar{j}-3\bar{k})\left[\frac{2\bar{i}+2\bar{j}-\bar{k}}{3}\right]$$

$$= \frac{12+4+3}{3} = \frac{19}{3}$$

The max. value of directional derivative $= |\nabla\phi| = \sqrt{(6)^2+(2)^2+(-3)^2}$

$$= \sqrt{49} = 7$$

Find the directional derivative of $F = 2xy+z^2$ at the point $(1,-1,3)$ in the direction of $\bar{i}+2\bar{j}+2\bar{k}$.

Given that the scalar point function $F = 2xy+z^2$

let $P(x,y,z) = P(1,-1,3)$ and vector $\vec{a} = \bar{i}+2\bar{j}+2\bar{k}$

$$\bar{n} = \nabla\phi = \bar{i}\frac{\partial F}{\partial x} + \bar{j}\frac{\partial F}{\partial y} + \bar{k}\frac{\partial F}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(2xy+z^2) + \bar{j}\frac{\partial}{\partial y}(2xy+z^2) + \bar{k}\frac{\partial}{\partial z}(2xy+z^2)$$

$$= \bar{i}(2y) + \bar{j}(2x) + \bar{k}(2z)$$

$$= 2y\bar{i} + 2x\bar{j} + 2z\bar{k}$$

$$\bar{n}_1\ (\nabla\phi)_{P(x,y,z)} = (\nabla\phi)_{(1,-1,3)} = -2\bar{i}+2\bar{j}+6\bar{k}$$

unit normal vector $\vec{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{(1)^2+(2)^2+(2)^2}} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{3}$

The directional derivative $\vec{E} = \nabla\phi.\vec{e} = \nabla\phi.\frac{\bar{a}}{|\bar{a}|}$

$$\vec{E} = (-2\bar{i}+2\bar{j}+6\bar{k})\frac{(\bar{i}+2\bar{j}+2\bar{k})}{3}$$

$$= \frac{-2+4+12}{3} = \frac{14}{3}$$

Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point P(3,-3, (1,2,-1) in the direction of PQ. where $Q = (3,-3,-2)$

Given that the scalar point function $\phi = x^2yz + 4xz^2$

let $P(x,y,z) = (1,2,-1)$

$$\bar{n} = \nabla\phi_1 = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2xyz + 4z^2) + \bar{j}(x^2z) + \bar{k}(x^2y + 8xz)$$

$$= (2xyz + 4z^2)\bar{i} + (x^2z)\bar{j} + (x^2y + 8xz)\bar{k}$$

$$\bar{n}_1 (\nabla\phi)_{P(x,y,z)} = (\nabla\phi)_{(1,2,-1)} = (-4+4)\bar{i} + (-1)\bar{j} + (2-8)\bar{k}$$

$$= -\bar{j} - 6\bar{k}$$

$\because \bar{a}$ is the vector in the direction of PQ $\vec{a} = \overline{OQ} - \overline{OP}$

$$\vec{a} = (3\bar{i} - 3\bar{j} - 2\bar{k}) - (\bar{i} + 2\bar{j} - \bar{k})$$

$$\vec{a} = 2\bar{i} - 5\bar{j} - \bar{k}$$

Unit Normal vector $\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\bar{i} - 5\bar{j} - \bar{k}}{\sqrt{(2)^2 + (-5)^2 + (-1)^2}} = \frac{2\bar{i} - 5\bar{j} - \bar{k}}{\sqrt{30}}$

$$\vec{E} = (\nabla\phi).\bar{e} = (\nabla\phi).\frac{\bar{a}}{|\bar{a}|}$$

$$= (-\bar{j} - 6\bar{k})\frac{(2\bar{i} - 5\bar{j} - \bar{k})}{\sqrt{30}}$$

$$= \frac{5+6}{\sqrt{30}} = \frac{11}{\sqrt{30}}$$

Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at the point P(2,1,-2) along the direction of the normal to the surface of $x^2 + y^2 + z^2 = 9$ along at the point (1,2,3)

Given that the scalar point function $\phi = 4xz^3 - 3x^2y^2z$

let $P(x,y,z) = (2,1,-2)$

$$\bar{n} = \nabla\phi_1 = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(4z^3 - 6xy^2z) + \bar{j}(0 - 6x^2yz) + \bar{k}(12xz^2 - 3x^2y^2)$$

$\bar{n}_1 (\nabla\phi)_{P(x,y,z)} = \nabla\phi_{(2,1,-2)} = \left(4(-2)^3 - 6(2)(1)(-2)\right)\bar{i} + \bar{j}\left(-6(2)^2(1)(-2)\right) + \bar{k}\left(12(2)(-2)^2 - 3(2)^2(1)^2\right)$

$= -8\bar{i} + 48\bar{j} + 84\bar{k}$

Another scalar point function $\phi_1 = x^2+y^2+z^2-9$ and

let $P(x,y,z) = (1,2,3)$

$$\bar{n} = \nabla\phi_1 = \bar{i}\frac{\partial\phi_1}{\partial x} + \bar{j}\frac{\partial\phi_1}{\partial y} + \bar{k}\frac{\partial\phi_1}{\partial z}$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$$

$$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\bar{a} = \bar{n}_1 (\nabla\phi_1)_{P(1,2,3)} = 2\bar{i} + 4\bar{j} + 6\bar{k}$$

unit normal vector $\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\bar{i}+4\bar{j}+6\bar{k}}{\sqrt{(2)^2+(4)^2+(6)^2}} = \frac{2\bar{i}+4\bar{j}+6\bar{k}}{\sqrt{56}}$

$$\bar{e} = (\nabla\phi)_{(2,1,-2)} \cdot \frac{\bar{a}}{|\bar{a}|} = (-8\bar{i}+48\bar{j}+84\bar{k}) \cdot \frac{(2\bar{i}+4\bar{j}+6\bar{k})}{\sqrt{56}}$$

$$= \frac{-16+192+504}{\sqrt{56}} = \frac{680}{\sqrt{56}}$$

Find the directional derivative of $\nabla(\nabla\phi)$ at the point $(1,-2,1)$ in the direction of normal to the surface $xy^2z = 3x+z^2$ where $\phi = 2x^3y^2z^4$.

$$P(x,y,z) = (1,-2,1)$$

$$xy^2z = 3x+z^2, \quad \phi = 2x^3y^2z^4$$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(2x^3y^2z^4)$$

$$= \bar{i}(2y^2z^4)(3x^2) + \bar{j}(2x^3z^4)2y + \bar{k}(2x^3y^2)(4z^3)$$

$$= (6x^2y^2z^4)\bar{i} + (4x^3yz^4)\bar{j} + (8x^3y^2z^3)\bar{k}$$

$$\nabla\cdot\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(6x^2y^2z^4) + \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(4x^3yz^4) + \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(8x^3y^2z^3)$$

$$I' = (12xy^2z^4 + 12x^2yz^4 + 24x^2y^2z^3)\bar{i} + (12x^2yz^4 + 4x^3z^4 + 16x^3yz^3)(\bar{j}) +$$
$$(24x^2y^2z^3 + 16x^3yz^3 + 24x^3y^2z^2)\bar{k}$$

PP

$$\nabla\cdot(\nabla\phi)_{(1,-2,1)} = \left(12(1)(-2)^2(1)^4 + 12(1)^2(-2)(1)^4 + 24(1)^2(-2)^2(1)^3\right) + \bar{i} + \left(12(1)^2(-2)(1)^4 + \right.$$
$$\left. 4(1)^3(1)^4 + 16(1)^3(-2)(1)^3\right)\bar{j} + \left(24(1)^2(-2)^2(1)^3 + 16(1)^3(-2)(1)^3 + 24(1)^3(-2)^2(1)^2\right)\bar{k}$$

$$= (48 - 24 + 96)\bar{i} + (-24 + 4 - 32)\bar{j} + (96 - 32 + 96)\bar{k}$$

$$= \bar{i}(12xy^2z^4) + \bar{j}\,(\overset{4x^3z^4}{\cancel{12x^2yz^4}}) + \bar{k}(24x^3y^2z^2)$$

$$(\nabla\phi)_{(1,-2,1)} = \bar{i}\,(12(1)(-2)^2(1)^4) + \bar{j}\,(4(1)^3\cancel{(-2)}(1)^4) + \bar{k}\,(24(1)^3(-2)^2(1)^2)$$

$$= 48\bar{i} + 4\bar{j} + 96\bar{k}$$

$$\phi_1 = xy^2z - 3x - z^2 \quad (1,-2,1)$$

$$\bar{a} = (\nabla\phi_1)_{(1,-2,1)} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x y^2 z - 3x - z^2)$$

$$= \bar{i}\,(y^2z) - 3) + \bar{j}(2xyz) + \bar{k}\,(xy^2 - 2z)$$

$$= \bar{i} - 4\bar{j} + 2\bar{k}$$

$$\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i} - 4\bar{j} + 2\bar{k}}{\sqrt{(1)^2 + (-4)^2 + (2)^2}} = \frac{\bar{i} - 4\bar{j} + 2\bar{k}}{\sqrt{21}}$$

$$\vec{E} = (48\bar{i} + 4\bar{j} + 96\bar{k})\left(\frac{\bar{i} - 4\bar{j} + 2\bar{k}}{\sqrt{21}}\right)$$

$$= \frac{48 - 16 + 192}{\sqrt{21}} = \frac{224}{\sqrt{21}}$$

Find the directional derivative of the $\phi = 1/\cancel{r}$ in the direction of $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ at the point $P(1,1,2)$

Given that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ — ① $\Rightarrow r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$ — ②

squ on b/s $r^2 = x^2 + y^2 + z^2$ — ③

Diff equ ③ partially w.r.t $x, y, \& z$

$$2r \cdot \frac{\partial r}{\partial x} = 2x \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \; ; \; \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla \phi = \left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right]\left(\frac{1}{r}\right)$$

$$= \bar{i}(-1)r^{-2}\frac{\partial r}{\partial x} + \bar{j}(-1)r^{-2}\frac{\partial r}{\partial y} + \bar{k}(-1)r^{-2}\frac{\partial r}{\partial z}$$

$$= -r^{-2}\left[\left(\frac{x}{r}\right)\bar{i} + \left(\frac{y}{r}\right)\bar{j} + \left(\frac{z}{r}\right)\bar{k}\right]$$

$$= \frac{-1}{r^3}\left[x\bar{i} + y\bar{j} + z\bar{k}\right] = \frac{-\bar{r}}{r^3} \quad \because \text{equ } ①$$

cubing on b/s of equ ②

$$r^3 = (x^2+y^2+z^2)^{3/2}$$

$$\nabla\phi = -\frac{[x\bar{i} + y\bar{j} + z\bar{k}]}{(x^2+y^2+z^2)^{3/2}}$$

$$(\nabla\phi)_{P(1,1,2)} = -\frac{[\bar{i}+\bar{j}+2\bar{k}]}{(6)^{3/2}}$$

$$\bar{a} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$|\bar{a}|_{P(1,1,2)} = \bar{i} + \bar{j} + 2\bar{k}$$

$$\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i}+\bar{j}+2\bar{k}}{\sqrt{6}}$$

$$\bar{e} = \nabla\phi \cdot \bar{e} = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|} = -\frac{[\bar{i}+\bar{j}+2\bar{k}]}{(6)^{3/2}} \cdot \frac{(\bar{i}+\bar{j}+2\bar{k})}{\sqrt{6}}$$

$$= -\frac{(1+1+4)}{\sqrt{216 \times 6}} = \frac{-6}{36} = \frac{-1}{6}$$

Find the directional derivative of $xy\phi = xy^2 + yz^2 + zx^2$ along the tangent to the curve $x=t$, $y=t^2$, $z=t^3$ at the point $(1,1,1)$

Given that $\phi = xy^2 + yz^2 + zx^2$

and the point $P(1,1,1)$

$$\nabla\phi_1 = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^2 + zx^2)$$

$$= \bar{i}(y^2+2xz)+\bar{j}(2xy+z^2)+\bar{k}(2yz+x^2)$$

$$(\nabla\phi)_{(1,1,1)} = [1+2(1)(1)]\,\bar{i}+(2(1)(1)+1)\,\bar{j}+(2(1)(1)+1)\,\bar{k}$$

$$= 3\bar{i}+3\bar{j}+3\bar{k}$$

$\bar{a}=?$ tangent to the curve $x=t$

$$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$$

$$= t\bar{i}+t^2\bar{j}+t^3\bar{k}$$

$$\bar{a} = \frac{\partial \bar{r}}{\partial t} = \bar{i}+2t\,j+3t^2\bar{k}$$

$$\bar{a}.\left(\frac{\partial \bar{r}}{\partial t}\right)_{(1,1,1)} = \bar{i}+2\bar{j}+3\bar{k}$$

$$\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{(1)^2+(2)^2+(3)^2}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$$

$$\vec{E} = (\nabla\phi).\bar{e} = (3\bar{i}+3\bar{j}+3\bar{k}).\left(\frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}\right)$$

$$= \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

Divergence of vector: If $\vec{F} = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$ then divergence of $\vec{F}$ is denoted by $div(\vec{F})$ and it is defined as

$$div(F) = \nabla.\vec{F} = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right).(f_1\bar{i}+f_2\bar{j}+f_3\bar{k})$$

$$= \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

Properties:

i. The divergence of a vector point function is scalar point function.

ii. $\nabla.\bar{F} \neq \bar{F}.\nabla$

iii. If $\bar{F}$ is a const. function divergence of $\bar{F}=0$ i.e $div(c)=0$

iv. If c is a const. and $\bar{F}$ is a vector then divergence (div) $(c\bar{F}) = c\, div(\bar{F})$.

v. If $div(\bar{F})=0$ then $\bar{F}$ is called solenoidal vector.

vi. The div($\bar{F}$) measures the outward flow (or) expansion of fluid from their point at any time.

vii. If div($\bar{F}$) = 0 there is no expansion

curl of vector point function: If $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then curl of vector is denoted by curl ($\bar{F}$) = $\nabla \times \bar{F}$ and it is defined as curl ($\bar{F}$) = $\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

properties:

i) curl of a vector point function is a vector quantity.

ii) $\nabla \times \bar{F} \neq \bar{F} \times \nabla$

iii) If $\bar{F}$ is a const. vector then curl ($\bar{F}$) = 0

iv) If curl ($\bar{F}$) = 0 then $\bar{F}$ is called irrotational vector.

v) curl ($c.\bar{F}$) = c curl ($\bar{F}$) if c is a const. and $\bar{F}$ is vector

vi) If $\bar{F}$ is irrotational then there will always exist a scalar function $\phi(x,y,z)$ such that $\bar{F}$ = (grad ϕ) = $\nabla\phi$ then. Here ϕ is called scalar potential function of $\bar{F}$.

vii) If $\nabla \times \bar{F} = 0$ iff there exist a scalar function ϕ such that $\bar{F}$ = (grad ϕ)

viii) using the curl we measure the circulation density of the fluid.

Find div ($\bar{F}$) & curl ($\bar{F}$) where $\bar{F} = xyz\bar{i} + 3x^2y\bar{j} + (xz^2 - y)\bar{k}$ at the point (2, −1, 1).

$$\bar{F} = \underset{f_1}{(xyz)}\bar{i} + \underset{f_2}{3x^2y}\bar{j} + \underset{f_3}{(xz^2 - y)}\bar{k}$$

$P(x,y,z) = (2,-1,1)$

$$\text{div.}\bar{F} = \nabla.\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}[xyz] + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}[xz^2-y]$$

$$= yz + 3x^2 + 2xz$$

$$\text{div.}\bar{F} = \nabla.\bar{F}_{(2,-1,1)} = (-1)(1) + 3(2)^2 + 2(2)(1) = 15$$

$$\text{curl } \bar{F} = \nabla\times\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\times(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \bar{i}[0-1-0] - \bar{j}[0-z^2-0-xy] + \bar{k}[6xy - xz]$$

$$= -\bar{i} - \bar{j}[z^2-xy] + \bar{k}[6xy - xz]$$

$$(\text{curl } \bar{F})_{(2,-1,1)} = -\bar{i} - \bar{j}[(1)^2 - 2(-1)] + \bar{k}[6(2)(-1) - 2(1)]$$

$$= -\bar{i} - 3\bar{j} - 14\bar{k}$$

Find div(F) & curl(F) where $\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

Given that $\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

$$\text{div } \bar{F} = \nabla.\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \left(\bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_2}{\partial y} + \bar{k}\frac{\partial f_3}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$= \bar{i}\underbrace{(3x^2-3yz)}_{f_1} + \bar{j}\underbrace{(3y^2-3xz)}_{f_2} + \bar{k}\underbrace{(3z^2-3xy)}_{f_3}$$

$$\text{curl } \bar{F} = \nabla\times\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\times(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] + \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \bar{i}\,[-8x + 8x] + \bar{j}\,(-8y + 8y) + \bar{k}\,(-8z + 8z)$$

$$= \bar{i}(0) + \bar{j}(0) + \bar{k}(0)$$

$$= 0$$

Prove that $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$ is solenoidal

$$\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$$

$$\text{div}.\bar{F} = \nabla.\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$\text{div}.\bar{F} = \nabla.\bar{F} = 0 + 0 + 0 = 0$$

Then $\bar{F}$ is called solenoidal vector.

If $V = (3x - 2y + z)\bar{i} + (4x + ay - z)\bar{j} + [x - y + 2z]\bar{k}$ then find the value of a.

$$V = (3x - 2y + z)\bar{i} + (4x + ay - z)\bar{j} + (x - y + 2z)\bar{k}$$

$$\text{curl } \bar{V} = \nabla \times \bar{V} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \underset{f_1}{(3x - 2y + z)} & \underset{f_2}{(4x + ay - z)} & \underset{f_3}{(x - y + 2z)} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \bar{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \bar{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$= \bar{i}[-1 + 1] - \bar{j}[1 - 1] + \bar{k}[4 + 2] = 0\bar{i}$$

If $\bar{v} = \dfrac{x\bar{i} + y\bar{j}}{x^2+y^2}$ is solenoidal vector.

$$\bar{v} = \frac{x\bar{i} + y\bar{j}}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2}\bar{i} + \frac{y}{x^2+y^2}\bar{j}$$

$$\text{div}.\bar{v} = \nabla.\bar{v} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_2}{\partial y} + \bar{k}\frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} \quad \because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= \frac{(x^2+y^2) - 2x^2 + (x^2+y^2) - 2y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-2x^2-2y^2}{(x^2+y^2)^2} = 0$$

prove that $\nabla.r^n.\bar{r} = \text{div}(r^n.\bar{r}) = (n+3)r^n$ (or) find grad $(r^n.\bar{r})$ and find 'n' value and for what value of n the given vector is solenoidal.

we know that $\bar{r} = \overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$ —①

$|\bar{r}| = \sqrt{x^2+y^2+z^2}$ —② $\Rightarrow r^2 = x^2+y^2+z^2$ —③

Diff equ ③ partially w.r.t x, y, z

$$\left.\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \\ 2r\frac{\partial r}{\partial y} &= 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} \\ 2r.\frac{\partial r}{\partial z} &= 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} \end{aligned}\right\} \text{—④}$$

$\nabla.(r^n.\bar{r}) = \text{div}(r^n.\bar{r})$

$$=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\left[r^n(x\bar{i}+y\bar{j}+z\bar{k})\right]$$

$$=\frac{\partial}{\partial x}[r^n.x]+\frac{\partial}{\partial y}[r^n.y]+\frac{\partial}{\partial z}[r^n.z]$$

$$=\left[1.r^n+x.nr^{n-1}\frac{\partial r}{\partial x}\right]+\left[1.r^n+y\,nr^{n-1}\frac{\partial r}{\partial y}\right]+\left[1.r^n+z\,nr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$=3r^n+nr^{n-1}\left[x.\frac{\partial r}{\partial x}+y\frac{\partial r}{\partial y}+z\frac{\partial r}{\partial z}\right]$$

$$=3r^n+nr^{n-1}\left[x.\frac{x}{r}+y.\frac{y}{r}+z.\frac{z}{r}\right] \quad \because \text{from eqn } ④$$

$$=3r^n+nr^{n-2}\left[x^2+y^2+z^2\right] \quad \because \text{from eqn } ③$$

$$=3r^n+nr^{n-2}[r^2]$$

$$=3r^n+nr^n=[3+n]\,r^n$$

prove that $\frac{\bar{r}}{r^3}$ is solenoidal $\nabla\cdot r^{-3}.\bar{r}=0$

we know that $\bar{r}=\bar{OP}=x\bar{i}+y\bar{j}+z\bar{k}$ — ①

$$|\bar{r}|=\sqrt{x^2+y^2+z^2}$$

$$r^2=x^2+y^2+z^2 \quad — ③$$

Diff eqn ③ partially w.r.t x, y, z

$$\left.\begin{aligned}
2r\frac{\partial r}{\partial x}&=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}\\
2r.\frac{\partial r}{\partial y}&=2y \Rightarrow \frac{\partial r}{\partial y}=\frac{y}{r}\\
2r.\frac{\partial r}{\partial z}&=2z \Rightarrow \frac{\partial r}{\partial z}=\frac{z}{r}
\end{aligned}\right\} — ④$$

$$\nabla(r^{-3}.\bar{r})=\text{div}\,(r^{-3}.\bar{r})$$

$$=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\left[r^{-3}(x\bar{i}+y\bar{j}+z\bar{k})\right]$$

$$=\frac{\partial}{\partial x}[r^{-3}.x]+\frac{\partial}{\partial y}[r^{-3}.y]+\frac{\partial}{\partial z}[r^{-3}z]$$

$$=\left[1.r^{-3}+x.(-3)r^{-3-1}\frac{\partial r}{\partial x}\right]+\left[1.r^{-3}+y(-3)r^{-3-1}\frac{\partial r}{\partial y}\right]+\left[1.r^{-3}+z(-3)r^{-3-1}\frac{\partial r}{\partial z}\right]$$

$$= 3r^{-3} - 3r^{-4}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3r^{-3} - 3r^{-4}\left[x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right]$$ from eqn (4)

$$= 3r^{-3} - 3r^{-5}\left[x^2+y^2+z^2\right]$$

$$= 3r^{-3} - 3r^{-5}\left[r^2\right]$$ from eqn (3)

$$= 3r^{-3} - 3r^{-3} = 0$$

Prove that $r^n.\bar{r}$ is irrotational.

$$\text{curl}(r^n.\bar{r}) = \nabla \times (r^n.\bar{r}) = 0$$

we know that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ — (1)

$|\bar{r}| = \sqrt{x^2+y^2+z^2}$ — (2) $\quad r^2 = x^2+y^2+z^2$ — (3)

Diff eqn (3) partially w.r.t x, y, z

$$\left.\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \\ 2r.\frac{\partial r}{\partial y} &= 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} \\ 2r.\frac{\partial r}{\partial z} &= 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} \end{aligned}\right\} \text{— (4)}$$

$$\nabla\times(r^n.\bar{r}) = \text{curl}(r^n.\bar{r})$$

$$= \left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right] \times \left[r^n(x\bar{i}+y\bar{j}+z\bar{k})\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n.x & r^n y & r^n.z \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] - \bar{j}\left[\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n.x)\right] + \bar{k}\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right]$$

$$= \bar{i}\left[\left(r^n(0) + z.nr^{n-1}\frac{\partial r}{\partial y}\right) - \left(r^n(0) + y.nr^{n-1}\frac{\partial r}{\partial z}\right)\right] -$$

$$\bar{j}\left[\left(r^n(0) + z.nr^{n-1}\frac{\partial r}{\partial x}\right) - \left(r^n(0) + x.nr^{n-1}\frac{\partial r}{\partial z}\right)\right] +$$

$$\bar{k}\left[\left(r^n(0)+y\,n r^{n-1}\frac{\partial r}{\partial x}\right)-\left(r^n(0)+x.n r^{n-1}\frac{\partial r}{\partial y}\right)\right]$$

$$=\left[\left(z.n r^{n-1}\frac{\partial r}{\partial y}\right)-\left(y n r^{n-1}\frac{\partial r}{\partial z}\right)\right]\bar{i}-\left[\left(z.n r^{n-1}\frac{\partial r}{\partial x}\right)\mp x.n r^{n-1}\frac{\partial r}{\partial z}\right]\bar{j}+$$

$$\left[\left(y n r^{n-1}\frac{\partial r}{\partial x}\right)-\left(x.n r^{n-1}\frac{\partial r}{\partial y}\right)\right]\bar{k}$$

pro $\Rightarrow$ $\left(z n.r^{n-1}\frac{y}{r}-y n r^{n-1}\frac{z}{r}\right)\bar{i}-\left(z n r^{n-1}\frac{x}{r}\mp x.n r^{n-1}\frac{z}{r}\right)\bar{j}+\bar{k}\left(y n r^{n-1}\frac{x}{r}-x n r^{n-1}\frac{y}{r}\right)$

$= n r^{n-2}[(zy-yz]\bar{i}-(xz\mp xz)\bar{j}+\bar{k}(xy-xy)] = n r^{n-2}[0]=\bar{0}$

prove that the vector $\bar{F}=[y^3-z^3+3yz-2x]\bar{i}+(3xz+2xy)\bar{j}$ $+[3xy-2xz+2z]\bar{k}$ is solenoidal and irrotational

$$\bar{F}=\underbrace{[y^3-z^3+3yz-2x]}_{f_1}\bar{i}+\underbrace{[3xz+2xy]}_{f_2}\bar{j}+\underbrace{[3xy-2xz+2z]}_{f_3}\bar{k}$$

$$\operatorname{div}(\bar{F})=\nabla.\bar{F}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

$$=-2+2x-2x+2=0$$

$\therefore \bar{F}$ is solenoidal vector

$$\operatorname{curl}(\bar{F})=\nabla\times\bar{F}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ [y^3-z^3+3yz-2x] & [3xz+2xy] & [3xy-2xz+2z]\end{vmatrix}$$

$$=\bar{i}\left[\frac{\partial}{\partial y}[3xy-2xz+2z]-\frac{\partial}{\partial z}[3xz+2xy]\right]-$$

$$\bar{j}\left[\frac{\partial}{\partial x}[3xy-2xz+2z]-\frac{\partial}{\partial z}[y^3-z^3+3yz-2x]\right]+$$

$$\bar{k}\left[\frac{\partial}{\partial x}[3xz+2xy]-\frac{\partial}{\partial y}[y^3-z^3+3yz-2x]\right]$$

$$=\bar{i}[3x-3x]-\bar{j}[3y-2+3z^2-3y]+\bar{k}[3z+2y\mp 3y^2-3z]$$

$= 2\bar{j}+3z$

$= 2+3z = [2-3z^2]\bar{j}+[2y-3y^2]\bar{k}$

The given vector is solenoidal not a irrotational.

Find the values of a and b such that the vector $\bar{F}$ is irrotational where $\bar{F}$ is $[2xy+3yz]\bar{i}+[x^2+azx-4z^2]\bar{j}+[3xy+2byz]\bar{k}$

$$\bar{F} = [2xy+3yz]\bar{i}+[x^2+azx-4z^2]\bar{j}+[3xy+2byz]\bar{k}$$

$$\nabla \times \bar{F} = \text{curl}(\bar{F}) = 0$$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ [2xy+3yz] & [x^2+azx-4z^2] & [3xy+2byz] \end{vmatrix} = 0$$

$$\Rightarrow \bar{i}\,[3x+2bz-(ax-8z)]-\bar{j}\,[3y-3y]+\bar{k}\,[2x+az-(2x+3z)]=\bar{0}$$

$$= \bar{i}\,[(3-a)x+(2b+8)z]+0+\bar{k}\,(a-3)z = 0\bar{i}+0\bar{j}+0\bar{k}$$

$$\Rightarrow (3-a)x+(2b+8)=0$$

$$a-3=0, \quad 2b+8=0$$

$$a=3, \quad b=-4$$

Find the values of a, b and c such that the vector $\bar{F}$ is irrotational when $\bar{F}=(x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ and find scalar potential function

$$\bar{F} = (x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$$

$$\nabla \times \bar{F} = \text{curl}(\bar{F}) = 0$$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$\Rightarrow \bar{i}\,[c+1]-\bar{j}\,[4-a]+\bar{k}\,[b-2]=0\bar{i}+0\bar{j}+0\bar{k}$$

$$c+1=0, \quad -4+a=0, \quad b-2=0$$

$$c=-1, \quad a=4, \quad b=2$$

Scalar ~~point~~ potential function $\bar{F} = (x+2y+4z)\bar{i}+(2x-3y-z)\bar{j}+(4x-y+2z)\bar{k}$

$\because \bar{F}$ is irrotational there is always exist a scalar potential function such that $\phi(x,y,z)$ $\bar{F}=\nabla\phi$

$(x+2y+4z)\bar{i}+(2x-3y-z)\bar{j}+(4x-y+2z)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = x+2y+4z$; $\frac{\partial\phi}{\partial y} = 2x-3y-z$; $\frac{\partial\phi}{\partial z} = 4x-y+2z$

Apply integration on bothsides

$\int\frac{\partial\phi}{\partial x} = \int x+2y+4z$; $\int\frac{\partial\phi}{\partial y} = \int 2x-3y-z$; $\int\frac{\partial\phi}{\partial z} = \int 4x-y+2z$

$\phi = \frac{x^2}{2}+2yx+4zx+c$; $\phi = 2xy-\frac{3y^2}{2}-zy+c$; $\phi = 4xz-yz+\frac{2z^2}{2}+c$

$\phi(x,y,z) = \frac{x^2}{2}-\frac{3y^2}{2}+z^2+2xy+2xy+4xz-yz+4xz-yz+c$

$\cancel{\left\{\frac{x^2}{2}-\frac{3y^2}{2}+z^2+4xy+8xz-2yz+\text{constant}\right\}}$

$\phi(x,y,z) = \frac{x^2}{2}-\frac{3y^2}{2}+z^2+2xy+4xz-yz+\text{constant}$

Show that the vector $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

$\bar{V} = (x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$

$\text{curl}(\bar{V}) = \nabla\times\bar{V} = 0$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-yz) & (y^2-zx) & (z^2-xy) \end{vmatrix} = \bar{0}$$

$= \bar{i}(-x-(-x))-\bar{j}(-y-(-y))+\bar{k}(-z-(-z)) = \bar{0}$

$= 0$

$\therefore \bar{V}$ is irrotational vector $\phi(x,y,z)$ such that $\bar{V}=\nabla\phi$

Here always exist a scalar potential function $\bar{V}=\nabla\phi$

$(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$

$\int\frac{\partial\phi}{\partial x} = \int x^2-yz$; $\int\frac{\partial\phi}{\partial y} = \int y^2-xz$; $\int\frac{\partial\phi}{\partial z} = \int(z^2-xy)$

$\phi = \frac{x^3}{3}-xyz+c$; $\phi = \frac{y^3}{3}-xyz+c$; $\phi = \frac{z^3}{3}-xyz+c$

$\phi(x,y,z) = \frac{x^3}{3}+\frac{y^3}{3}+\frac{z^3}{3}-xyz+\text{constant}$

check whether the function $(x^2-y^3)\bar{i}+(y^2-3x)\bar{j}+(z^2-xy)\bar{k}$ is irrotation and hence find the scalar potential function corr. to it.

$$\bar{F}=(x^2-y^3)\bar{i}+(y^2-3x)\bar{j}+(z^2-xy)\bar{k}$$

$$\text{curl}(\bar{F}) = \nabla\times\bar{F}=0$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^3) & (y^2-3x) & (z^2-xy) \end{vmatrix}$$

$$=\bar{i}(-x-0)-\bar{j}(-y-0)+\bar{k}(-3+3y^2)$$

$$=-x\bar{i}+y\bar{j}+(-3+3y^2)\bar{k}$$

$\therefore \nabla\times\bar{F}$ (or) $\text{curl}(\bar{F})\neq 0 \therefore \bar{F}$ is not irrotational vector and it is not possible to find the scalar potential function.

Laplacian operator: It is denoted by '∇^2' and it is defined as $\nabla^2=\nabla.\nabla=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)$

$$=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$\nabla^2(\phi)=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

If $\nabla\phi=0$ the ϕ is said to be Laplacian equ. and this ϕ is called Harmonic function.

Prove that div(grad r^n) or $\nabla^2(r^2)=n(n+1)r^{n-2}$

we know that $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$ and $|\bar{r}|=\sqrt{x^2+y^2+z^2}$ — ②

$$r^2=x^2+y^2+z^2 \quad — ①$$

— ③

Diffn 'r^2' w.r.t partially w.r.t 'x, y, z'

$$\left.\begin{aligned} 2r\frac{\partial r}{\partial x}&=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r} \\ 2r.\frac{\partial r}{\partial z}&=2z \Rightarrow \frac{\partial r}{\partial z}=\frac{z}{r} \end{aligned}\right\} \quad 2r.\frac{\partial r}{\partial y}=2y \Rightarrow \frac{\partial r}{\partial y}=\frac{y}{r}$$

— ④

L.H.S $= \text{div}(\text{grad } r^n) = \nabla^2(r^n) = \nabla(\nabla r^n)$

$$= \nabla\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) r^n$$

$$= \nabla\left[\bar{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \bar{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \bar{k}\, n r^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= \nabla\left[\bar{i}\, n r^{n-1}\frac{x}{r} + \bar{j}\, n r^{n-1}\frac{y}{r} + \bar{k}\, n r^{n-1}\frac{z}{r}\right] \quad \because \text{from } (4)$$

$$= \nabla\left[\bar{i}\, n r^{n-2} x + \bar{j}\, n r^{n-2} y + \bar{k}\, n r^{n-2} z\right]$$

$$= n\left[\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left[(x r^{n-2})\bar{i} + (y r^{n-2})\bar{j} + (z r^{n-2})\bar{k}\right]\right]$$

$$= n\left[\frac{\partial}{\partial x}(x r^{n-2}) + \frac{\partial}{\partial y}(y r^{n-2}) + \frac{\partial}{\partial z}(z \cdot r^{n-2})\right]$$

$$= n\left[1 \cdot r^{n-2} + x(n-2) r^{n-3}\frac{\partial r}{\partial x} + 1 \cdot r^{n-2} + y(n-2) r^{n-3}\frac{\partial r}{\partial y} + 1 \cdot r^{n-2} + z(n-2) r^{n-3}\frac{\partial r}{\partial z}\right]$$

$$= n\left[3 r^{n-2} + x(n-2) r^{n-3}\frac{x}{r} + y(n-2) r^{n-3}\frac{y}{r} + z(n-2) r^{n-3}\frac{z}{r}\right]$$

$$= n\left[3 r^{n-2} + (n-2) r^{n-3}\left[\frac{x^2+y^2+z^2}{r}\right]\right]$$

$$= n\left[3 r^{n-2} + (n-2) r^{n-3}\left[\frac{r^2}{r}\right]\right] = n\left[3 r^{n-2} + (n-2) r^{n-2}\right]$$

L.H.S $= n[3+n-2]\, r^{n-2}$

$= n[n+1]\, r^{n-2}$

LHS = RHS

prove that $\nabla^2(\log r) = \frac{1}{r^2}$

we know that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ — (1) and $|\bar{r}| = \sqrt{x^2+y^2+z^2}$ — (2)

$$r^2 = x^2+y^2+z^2 \quad (3)$$

Diff eqn (3) partially w.r.t x, y, z

$$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}; \quad 2r \cdot \frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r \cdot \frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} \quad — (4)$$

$$\text{L.H.S} = \nabla\,[\nabla \log r]$$

$$= \nabla\left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right]\log r$$

$$= \nabla\left[\bar{i}\cdot\frac{1}{r}\cdot\frac{\partial r}{\partial x} + \bar{j}\,\frac{1}{r}\cdot\frac{\partial r}{\partial y} + \bar{k}\,\frac{1}{r}\cdot\frac{\partial r}{\partial z}\right]$$

$$= \nabla\left[\bar{i}\,\frac{1}{r}\cdot\frac{x}{r} + \bar{j}\,\frac{1}{r}\cdot\frac{y}{r} + \bar{k}\cdot\frac{1}{r}\cdot\frac{z}{r}\right]$$

$$= \nabla\left[\bar{i}\;r^{-2}\cdot x + \bar{j}\cdot r^{-2}\cdot y + \bar{k}\cdot r^{-2}\cdot z\right]$$

$$= \left[\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left[(x r^{-2})\bar{i} + (y\cdot r^{n-2})\bar{j} + (z\cdot r^{n-2})\bar{k}\right]\right]$$

$$= \left[\frac{\partial}{\partial x}(x r^{-2}) + \frac{\partial}{\partial y}(y r^{-2}) + \frac{\partial}{\partial z}(z\cdot r^{-2})\right]$$

$$= \left[1\cdot r^{-2} + x(-2)\cdot r^{-3}\cdot\frac{\partial r}{\partial x} + r^{-2} + y(-2)\cdot r^{-3}\cdot\frac{\partial r}{\partial y} + r^{-2} + z(-2)\cdot r^{-3}\cdot\frac{\partial r}{\partial z}\right]$$

$$= \left[3r^{-2} + (-2)r^{-3})\cdot\left[\frac{x^2+y^2+z^2}{r}\right]\right]$$

$$= \left[3r^{-2} - 2r^{-3}\cdot\left[\frac{r^2}{r}\right]\right] = \left[3r^{-2} - 2r^{-2}\right] = r^{-2} = \frac{1}{r^2} = \text{L.H.S}$$

Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$

We know that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ — (1)

$|\bar{r}| = \sqrt{x^2+y^2+z^2}$ — (2) and

$r^2 = x^2+y^2+z^2$ — (3)

Diff equ (3) partially w.r.t x, y, z

$$\left.\begin{aligned} 2r\cdot\frac{\partial r}{\partial x} &= 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}\\ 2r\cdot\frac{\partial r}{\partial y} &= 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}\\ 2r\cdot\frac{\partial r}{\partial z} &= 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}\end{aligned}\right\} \text{— (4)}$$

$$\text{L.H.S} = \nabla\left(\nabla\frac{1}{r}\right)$$

$$= \nabla \left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right]\left(\frac{1}{r}\right)$$

$$= \nabla \left[\bar{i}\left(\frac{-1}{r^2}\right)\cdot\frac{\partial r}{\partial x} + \bar{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \bar{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}\right]$$

$$= \nabla \left[\bar{i}\,\frac{-1}{r^2}\cdot\frac{x}{r} + \bar{j}\,\frac{-1}{r^2}\cdot\frac{y}{r} + \bar{k}\,\frac{-1}{r^2}\cdot\frac{z}{r}\right]$$

$$= \nabla\left[\bar{i} - r^{-3}\cdot x + \bar{j} - r^{-3}\cdot y + \bar{k} - r^{-3}\cdot z\right]$$

$$= \left[\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left[(x(-r^{-3}))\bar{i} + (y(-r^{-3}))\bar{j} + (z(-r^{-3}))\bar{k}\right]\right]$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left[\frac{\partial}{\partial x}(-x\cdot r^{-3}) + \frac{\partial}{\partial y}(-y\cdot r^{-3}) + \frac{\partial}{\partial z}(-z\cdot r^{-3})\right]$$

$$= \left[1.(-r^{-3}) + x.(3)r^{-4}\frac{\partial r}{\partial x} + 1(-r^{-3}) + y(3)r^{-4}\cdot\frac{\partial r}{\partial y} + 1.(-r^{-3}) + z.(3)r^{-4}\frac{\partial r}{\partial z}\right]$$

$$= -3r^{-3} + \left[3.r^{-4}\cdot\left(\frac{x^2+y^2+z^2}{r}\right)\right]$$

$$= -3r^{-3} + 3r^{-4}\cdot\left(\frac{r^2}{r}\right)$$

$$= -3r^{-3} + 3r^{-3} = 0 = \text{L.H.S}$$

prove that $\text{div}[\text{grad} f(r)] = \nabla(\nabla f(r)) = \nabla^2[f(r)] = f''(r) + \frac{2}{r}f'(r)$

we know that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ — ① and

$|\bar{r}| = \sqrt{x^2+y^2+z^2}$ — ② $\qquad r^2 = x^2+y^2+z^2$ — ③

Diff eqn ③ partially w.r.t x, y, z

$$2r.\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r.\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r.\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{— ④}$$

$$\text{L.H.S} = \text{div}[\text{grad} f(r)] = \nabla.[\nabla f(r)] = \nabla^2.[f(r)]$$

$$= \nabla\left[\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right] f(r)$$

$$= \nabla\left[\bar{i}\, f'(r)\frac{\partial r}{\partial x} + \bar{j}\, f'(r)\frac{\partial r}{\partial y} + \bar{k}\, f'(r)\frac{\partial r}{\partial z}\right]$$

$$= \nabla\left[\bar{i}\, f'(r).\frac{x}{r} + \bar{j}\, f'(r).\frac{y}{r} + \bar{k}\, f'(r).\frac{z}{r}\right]$$

$$= \left[\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left(x.\frac{f'(r)}{r}\bar{i} + y.\frac{f'(r)}{r}\bar{j} + z\frac{f'(r)}{r}\bar{k}\right)\right]$$

$$= \frac{\partial}{\partial x}\left[x\frac{f'(r)}{r}\right] + \frac{\partial}{\partial y}\left[y.\frac{f'(r)}{r}\right] + \frac{\partial}{\partial z}\left[z.\frac{f'(r)}{r}\right]$$

$$= 1.\frac{f'(r)}{r} + x\left[\frac{\partial}{\partial x}\frac{f'(r)}{r}\right] + 1\frac{f'(r)}{r} + y\left[\frac{\partial}{\partial y}\frac{f'(r)}{r}\right] + 1\frac{f'(r)}{r} + z\left[\frac{\partial}{\partial z}.\frac{f'(r)}{r}\right]$$

$$= \frac{3.f'(r)}{r} + x\left[\frac{rf''(r)\frac{\partial r}{\partial x} - f'(r)\frac{\partial r}{\partial x}}{r^2}\right] + y\left[\frac{rf''(r)\frac{\partial r}{\partial y} - f'(r)\frac{\partial r}{\partial y}}{r^2}\right] + z\left[\frac{rf''(r)\frac{\partial r}{\partial z} - f'(r)\frac{\partial r}{\partial z}}{r^2}\right]$$

$$= \frac{3f'(r)}{r} + x.\left[\frac{r.f''(r).\frac{x}{r} - f'(r).\frac{x}{r}}{r^2}\right] + y\left[\frac{rf''(r).\frac{y}{r} - f'(r).\frac{y}{r}}{r^2}\right] + z\left[\frac{rf''(r)\frac{z}{r} - f'(r).\frac{z}{r}}{r^2}\right]$$

$$L.H.S = \frac{3f'(r)}{r} + \frac{1}{r^2}\left[x^2f''(r) - x^2\frac{f'(r)}{r}\right] + \frac{1}{r^2}\left[y^2f''(r) - y^2.\frac{f'(r)}{r}\right] + \frac{1}{z^2}\left[z^2f''(r) - z^2\frac{f'(r)}{r}\right]$$

$$= \frac{3f'(r)}{r} + \frac{1}{r^2}f''(r)\left[\left[x^2+y^2+z^2\right] - \frac{f'(r)}{r}.\frac{1}{r^2}\left[x^2+y^2+z^2\right]\right]$$

$$= \frac{3f'(r)}{r} + \frac{1}{\cancel{r^2}}f''(r).\cancel{r^2} - \frac{f'(r)}{r}.\frac{1}{\cancel{r^2}}.\cancel{r^2}$$

$$= f''(r) + \frac{2f'(r)}{r} = RHS$$

Vector Identities

Vector differential operator (∇):

* $\text{grad}\,\phi = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = \Sigma\bar{i}\frac{\partial\phi}{\partial x}$

* $\text{div}.\bar{a} = \nabla.\bar{a} = \bar{i}\frac{\partial\bar{a}}{\partial x} + \bar{j}\frac{\partial\bar{a}}{\partial y} + \bar{k}\frac{\partial\bar{a}}{\partial z} = \Sigma\bar{i}\frac{\partial\bar{a}}{\partial x}$

* $\text{curl}(\bar{a}) = \nabla\times\bar{a} = \bar{i}\frac{\partial\bar{a}}{\partial x} + \bar{j}\frac{\partial\bar{a}}{\partial y} + \bar{k}\frac{\partial\bar{a}}{\partial z} = \Sigma\bar{i}\frac{\partial\bar{a}}{\partial x}$

scalar differential Eqn's:

$\rightarrow (\nabla.\bar{a}) = \bar{a}\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) = \Sigma(\bar{a}.\bar{i}).\frac{\partial}{\partial x}$

$\rightarrow (\bar{a}.\nabla)\bar{b} = \Sigma(\bar{a}.\bar{i})\frac{\partial\bar{b}}{\partial x}$ $\quad * (\bar{a}.\nabla).\phi = \Sigma(\bar{a}.\bar{i})\frac{\partial\phi}{\partial x}$

$\rightarrow (\bar{a}\times\nabla) = \bar{a}\times\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) = \Sigma(\bar{a}\times\bar{i})\frac{\partial}{\partial x}$

$\rightarrow (\bar{a}\times\nabla)\phi = \Sigma(\bar{a}\times\bar{i})\frac{\partial\phi}{\partial x}$

$\rightarrow (\bar{a}\times\nabla)\bar{b} = \Sigma(\bar{a}\times\bar{i})\frac{\partial\bar{b}}{\partial x}$

$\rightarrow \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$

$\rightarrow \bar{a}.(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b}).\bar{c}$

$= -(\bar{a}\times\bar{c}).\bar{b}$

$= -\bar{a}.(\bar{c}\times\bar{b})$

Vector Identities:

Theorem-1: prove that curl (grad ϕ) = 0

curl (grad ϕ) = 0

$\nabla(\nabla.\phi) = 0$

$$\nabla.\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\times(\nabla.\phi) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\times\left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right)$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$\Rightarrow \bar{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right] - \bar{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \bar{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right]$$

$= \bar{0}$

Theorem - 2: prove that $div(curl \bar{F}) = \bar{0}$

$$div(curl\ \bar{F}) = \bar{0}$$

$$\nabla(\nabla \times \bar{F}) = \bar{0}$$

$$\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \bar{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \bar{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$\nabla.(\nabla \times \bar{F}) = div(curl\ \bar{F}) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left[\bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right]$$

$$\Rightarrow \left[\frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z}\right] - \left[\frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_1}{\partial y \partial z}\right] + \left[\frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_1}{\partial y \partial z}\right]$$

$$\Rightarrow \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial x \partial y} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_1}{\partial y \partial z} = \bar{0}$$

Theorem - 3: prove that $div(f\nabla g) = f.\nabla^2 g + \nabla f.\nabla g$

let f, g are functions of $f(x,y,z)$ and $g(x,y,z)$

$$div(f\nabla g) = f.\nabla^2 g + \nabla f.\nabla g$$

$$\nabla(f\nabla g) = f.\nabla^2 g + \nabla f.\nabla g$$

$$(f\nabla g) = \bar{i}f.\frac{\partial g}{\partial x} + \bar{j}f\frac{\partial g}{\partial y} + \bar{k}f.\frac{\partial g}{\partial z}$$

$$\nabla(f\nabla g) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left(\bar{i}.f.\frac{\partial g}{\partial x} + \bar{j}f\frac{\partial g}{\partial y} + \bar{k}f.\frac{\partial g}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(f.\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f.\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f.\frac{\partial g}{\partial z}\right)$$

$\Rightarrow$ Diff partially w.r.t 'x, y, z

$$= f.\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}.\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}.\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}.\frac{\partial g}{\partial z}$$

$$\nabla(f\nabla g) = f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z}$$

$$= f.\nabla^2 g+\left(\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z}\right)\left(\bar{i}\frac{\partial g}{\partial x}+\bar{j}\frac{\partial g}{\partial y}+\bar{k}\frac{\partial g}{\partial z}\right)$$

$$= f.\nabla^2 g+\nabla f.\nabla g$$

Theorem-4: If $\bar{a}$ is differentiable function & ϕ is differentiable scalar function then prove that div$(\phi.\bar{a})$ =

div$(\phi.\bar{a})$ = grad $\phi.\bar{a}+\phi$. div. $\bar{a}$

$$\text{div}(\phi.\bar{a}) = \text{grad}\,\phi.\bar{a}+\phi.\text{div}.\bar{a}$$

$$\nabla(\phi\bar{a}) = \nabla\phi.\bar{a}+\phi.\nabla\bar{a}$$

$$\text{div}(\phi\bar{a}) = \nabla(\phi\bar{a}) = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(\phi\bar{a})$$

$$= \sum \bar{i}\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \sum \bar{i}\left[\frac{\partial\phi}{\partial x}\bar{a}+\phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum \bar{i}\frac{\partial\phi}{\partial x}.\bar{a}+\sum\bar{i}\phi\frac{\partial\bar{a}}{\partial x}$$

$$= \sum \bar{i}\frac{\partial\phi}{\partial x}.\bar{a}+\phi\sum\bar{i}.\frac{\partial\bar{a}}{\partial x}$$

$$= \nabla\phi.\bar{a}+\phi\nabla.\bar{a}$$

$$= \text{grad}\,\phi.\bar{a}+\phi(\text{div}.\bar{a})$$

Theorem-5: prove that curl$(\phi\bar{a})$ = grad $\phi\times\bar{a}+\phi$.curl$(\bar{a})$

$$\text{curl}(\phi\bar{a}) = \text{grad}\,\phi\times\bar{a}+\phi.\text{curl}(\bar{a})$$

$$\nabla\times(\phi\bar{a}) = \nabla\phi\times\bar{a}+\phi.(\nabla\times\bar{a})$$

$$\text{curl}(\phi.\bar{a}) = \nabla\times(\phi\bar{a}) = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\times(\phi\bar{a})$$

$$= \sum\bar{i}\frac{\partial}{\partial x}\times(\phi\bar{a})$$

$$= \sum\bar{i}\left[\frac{\partial\phi}{\partial x}\times\bar{a}+\phi.\times\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \Sigma \bar{i} \frac{\partial \phi}{\partial x} \times \bar{a} + \Sigma \bar{i} . \phi \times \frac{\partial \bar{a}}{\partial x}$$

$$= \Sigma \bar{i} \frac{\partial \phi}{\partial x} \times \bar{a} + \phi \, \Sigma \bar{i} \times \frac{\partial \bar{a}}{\partial x}$$

$$= \nabla \phi \times \bar{a} + \phi (\nabla \times \bar{a})$$

$$= \text{grad } \phi \times \bar{a} + \phi . \text{curl}(\bar{a})$$

**
Theorem - 6 : prove that div $(\bar{a} \times \bar{b}) = \bar{b} . \text{curl} \, \bar{a} - \bar{a} \, \text{curl} \, \bar{b}$

$$\text{div}(\bar{a} \times \bar{b}) = \bar{b} . \text{curl } \bar{a} - \bar{a} \text{ curl } \bar{b}$$

$$\nabla (\bar{a} \times \bar{b}) = \bar{b} . (\nabla \times \bar{a}) - \bar{a} . (\nabla \times \bar{b})$$

$$\text{LHS} = \text{div}(\bar{a} \times \bar{b}) = \nabla (\bar{a} \times \bar{b})$$

$$= \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) . (\bar{a} \times \bar{b})$$

$$= \Sigma \bar{i} \frac{\partial}{\partial x} (\bar{a} \times \bar{b})$$

$$= \Sigma \bar{i} \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right)$$

$$= \Sigma \bar{i} \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} \right) + \Sigma \bar{i} \left(\bar{a} \times \frac{\partial \bar{b}}{\partial x} \right)$$

$$\bar{a} . (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) . \bar{c} \quad (\text{or}) - (\bar{a} \times \bar{c}) . \bar{b}$$

$$\text{LHS} = \Sigma \bar{i} \times \frac{\partial \bar{a}}{\partial x} . \bar{b} - \Sigma \left(\bar{i} \times \frac{\partial \bar{b}}{\partial x} \right) . \bar{a}$$

$$= (\nabla \times \bar{a}) . \bar{b} - (\nabla \times \bar{b}) . \bar{a}$$

$$= \bar{b} . (\nabla \times \bar{a}) - \bar{a} (\nabla \times \bar{b})$$

$$= \bar{b} . \text{curl } \bar{a} - \bar{a} \text{ curl } \bar{b}$$

**
Theorem - 7 : prove that grad $(\bar{a} . \bar{b}) = (\bar{b} . \nabla) \bar{a} + (\bar{a} . \nabla) \bar{b} + (\bar{a} \times \text{curl } \bar{b}) + (\bar{b} \times \text{curl } \bar{a})$

$$\text{grad}(\bar{a} . \bar{b}) = (\bar{b} . \nabla) \bar{a} + (\bar{a} . \nabla) \bar{b} + (\bar{a} \times \text{curl } \bar{b}) + (\bar{b} \times \text{curl } \bar{a})$$

consider $\bar{a} \times (\text{curl } \bar{b}) = \bar{a} \times (\nabla \times \bar{b})$

$$= \bar{a} \times \Sigma \bar{i} \times \frac{\partial \bar{b}}{\partial x}$$

$$= \Sigma \bar{a} \times \left(\bar{i} \times \frac{\partial \bar{b}}{\partial x} \right)$$

$$\left[\because \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}\right]$$

$$= \Sigma\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)\bar{i} - \left((\bar{a}.\bar{i})\frac{\partial \bar{b}}{\partial x}\right)$$

$$\cancel{\left\{(\bar{a}.\nabla)\bar{b} = \Sigma(\bar{a}.\bar{i})\frac{\partial \bar{b}}{\partial x}\right\}}$$

$$(\bar{a}\times(\nabla\times\bar{b})) = \Sigma\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)\bar{i} - (\bar{a}.\nabla)\bar{b} \quad \text{——} ①$$

$$\rightarrow (\bar{b}\times \text{curl}\,\bar{a}) = \bar{b}\times(\nabla\times\bar{a})$$

$$= \bar{b}\times(\nabla\times\bar{a})$$

$$= \bar{b}\times\Sigma\,\bar{i}\times\frac{\partial \bar{a}}{\partial x}$$

$$= \Sigma\,\bar{b}\times\left(\bar{i}\times\frac{\partial \bar{a}}{\partial x}\right)$$

$$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$$

$$\Rightarrow \Sigma\left(\bar{b}.\frac{\partial \bar{a}}{\partial x}\right).\bar{i} - (\bar{b}.\bar{i})\frac{\partial \bar{a}}{\partial x}$$

$$\Rightarrow \Sigma\left(\bar{b}.\frac{\partial \bar{a}}{\partial x}\right)\bar{i} - (\bar{b}.\bar{i})\frac{\partial \bar{a}}{\partial x}$$

$$\Sigma\left(\bar{b}.\frac{\partial \bar{a}}{\partial x}\,\bar{i}\right) - (\bar{b}.\bar{i})\frac{\partial \bar{a}}{\partial x} \quad \text{——} ②$$

Adding equ ① & ②

$$(\bar{a}\times \text{curl}\,\bar{b}) + (\bar{b}\times\text{curl}\,\bar{a}) = \Sigma\left(\bar{b}.\frac{\partial \bar{a}}{\partial x}.\bar{i}\right) + \Sigma\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}.\bar{i}\right) - (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

Substitute in the equ

$$\text{grad}\,(\bar{a}.\bar{b}) = (\bar{b}.\nabla)\bar{a} + (\bar{a}.\nabla)\bar{b} + \Sigma\,\bar{i}\left(\bar{b}.\frac{\partial \bar{a}}{\partial x} + \bar{a}\frac{\partial \bar{b}}{\partial x}\right) - (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$\Rightarrow \Sigma\,\bar{i}\frac{\partial}{\partial x}(\bar{a}.\bar{b}) - (\bar{a}.\nabla)\bar{b} - (\bar{b}.\nabla)\bar{a}$$

$$\Rightarrow \text{grad}\,(\bar{a}.\bar{b}) - (\bar{a}.\nabla)\bar{b} - (\bar{b}.\nabla)\bar{a}$$

Theorem – 8: prove that $\text{curl}\,(\bar{a}\times\bar{b}) = \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$

$$\text{curl}\,(\bar{a}\times\bar{b}) = \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$\nabla \times (\bar{a} \times \bar{b}) = \bar{a}.(\nabla \bar{b}) - \bar{b}(\nabla \bar{a}) + \bar{b}.\nabla \bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$LHS = curl(\bar{a} \times \bar{b}) = \nabla \times (\bar{a} \times \bar{b})$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (\bar{a} \times \bar{b})$$

$$= \Sigma\, \bar{i} \times \frac{\partial}{\partial x}(\bar{a} \times \bar{b})$$

$$= \Sigma\, \bar{i} \times \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x}\right)$$

$$= \Sigma\, \bar{i} \times \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b}\right) + \Sigma\, \bar{i} \times \left(\bar{a} \times \frac{\partial \bar{b}}{\partial x}\right)$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$$

$$= \Sigma\left[(\bar{i}.\bar{b})\frac{\partial \bar{a}}{\partial x} - \left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right).\bar{b}\right] + \Sigma\left[\left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - (\bar{i}.\bar{a})\frac{\partial \bar{b}}{\partial x}\right]$$

$$= \Sigma\,(\bar{b}.\bar{i})\frac{\partial \bar{a}}{\partial x} - \Sigma\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b} + \Sigma\left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - \Sigma\,(\bar{i}.\bar{a})\frac{\partial \bar{b}}{\partial x}$$

$$= \bar{b}\,\Sigma\,\bar{i}\frac{\partial \bar{a}}{\partial x} - \bar{b}\,\Sigma\,\bar{i}\frac{\partial \bar{a}}{\partial x} + \bar{a}\,\Sigma\,\bar{i}\frac{\partial \bar{b}}{\partial x} - \bar{a}\,\Sigma\,\bar{i}\frac{\partial \bar{b}}{\partial x}$$

$$\cancel{= \bar{b}(\nabla.\bar{a}) - \bar{b}}$$

$$= (\bar{b}.\nabla)\bar{a} - (\nabla.\bar{a})\bar{b} + (\nabla.\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= (\nabla.\bar{b})\bar{a} - (\nabla.\bar{a})\bar{b} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

Theorem - 9: prove that $\nabla \times (\nabla \times \bar{a}) = \nabla(\nabla.\bar{a}) - \nabla^2 \bar{a}$

$$\nabla \times (\nabla \times \bar{a}) = \nabla(\nabla.\bar{a}) - \nabla^2 \bar{a}$$

$$LHS = \nabla \times (\nabla \times \bar{a}) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times \left(\bar{i}\frac{\partial \bar{a}}{\partial x} + \bar{j}\frac{\partial \bar{a}}{\partial y} + \bar{k}\frac{\partial \bar{a}}{\partial z}\right)$$

$$= \Sigma\,\bar{i} \times \frac{\partial}{\partial x}(\nabla \times \bar{a})$$

$$\bar{i} \times \frac{\partial}{\partial x}(\nabla \times \bar{a}) = \bar{i} \times \frac{\partial}{\partial x}\left[\bar{i} \times \frac{\partial \bar{a}}{\partial x} + \bar{j} \times \frac{\partial \bar{a}}{\partial y} + \bar{k} \times \frac{\partial \bar{a}}{\partial z}\right]$$

$$= \bar{i} \times \left[\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2} + \bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y} + \bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z}\right]$$

$$= \bar{i} \times \left(\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2}\right) + \bar{i} \times \left(\bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y}\right) + \bar{i} \times \left(\bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z}\right)$$

$$\Rightarrow \left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x^2}\right)\bar{i} - \frac{\partial^2\bar{a}}{\partial x^2} + \left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x\partial y}\right)\bar{j} + \left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k}$$

$$\because \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$

$$= \bar{i}\frac{\partial}{\partial x}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) + \bar{j}\frac{\partial}{\partial y}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) + \bar{k}\frac{\partial}{\partial z}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) - \frac{\partial^2\bar{a}}{\partial x^2}$$

$$= \left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right)\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) - \frac{\partial^2\bar{a}}{\partial x^2}$$

$$= \nabla\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) - \frac{\partial^2\bar{a}}{\partial x^2}$$

$$\Rightarrow \bar{i}\times\frac{\partial}{\partial x}(\nabla\times\bar{a}) = \nabla\sum\bar{i}\frac{\partial\bar{a}}{\partial x} - \sum\frac{\partial^2\bar{a}}{\partial x^2}$$

$$= \nabla(\nabla\cdot\bar{a}) - \left(\frac{\partial^2\bar{a}}{\partial x^2} + \frac{\partial^2\bar{a}}{\partial y^2} + \frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$\Rightarrow \nabla\times(\nabla\times\bar{a}) = \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a} \quad (\text{or})$$

$$\text{curl curl } \bar{a} = \text{grad div } \bar{a} - \nabla^2\bar{a}$$

prove that $(\nabla f\times\nabla g)$ is solenoidal

$(\nabla f\times\nabla g)$ is solenoidal

from Theorem - 6 · $\text{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\text{curl }\bar{a} - \bar{a}\text{ curl }\bar{b}$

let $\bar{a} = \nabla f$ and $\bar{b} = \nabla g$

Then $\text{div}(\nabla f\times\nabla g) = \nabla g\cdot\text{curl}(\nabla f) - \nabla f\cdot\text{curl}(\nabla g)$

$\because \text{curl}((\nabla f)\cdot(\nabla g)) = 0$

$\text{div}(\nabla f\times\nabla g) = \nabla g\cdot\text{curl}(\nabla f) - \nabla f\cdot\text{curl}(\nabla g) = 0$

$\therefore \nabla f\times\nabla g$ is solenoidal